Differential Geometry I $\boxed{\text{Week 9}}$

Let $M \subseteq \mathbb{R}^n$ be a submanifold, $p, q \in M$ (M of class C^1)

i) <u>Extrinsic distance</u>: $\|p-q\|$

ii) <u>Intrinsic distance</u>: If M is connected, then

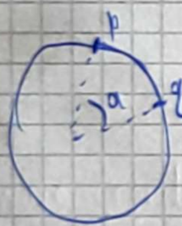
$$d_M(p,q) = \inf\left\{ \ell(\gamma) \mid \gamma : [a,b] \to M \text{ is piecewise } C^1 \text{ with } \gamma(a) = p \text{ and } \gamma(b) = q \right\}$$

(So distance when restricted to move only along M)

Trivially $d_M(p,q) \geq \|p-q\|$.

<u>Definition</u>: Let $M_1 \subseteq \mathbb{R}^n$, $M_2 \subseteq \mathbb{R}^k$ be submanifolds of class C^1. We will say that they are (intrinsically) <u>isometric</u> if there exists a bijective $f: M_1 \to M_2$ such that $d_{M_2}(f(p), f(q)) = d_{M_1}(p,q)$. Such an f: isometry

<u>Example</u>: On the sphere $\mathbb{S}^2$ of radius 1.



$d_{\mathbb{S}^2}(p,q) = a$ (angle between p, q)

$\|p-q\| = 2\sin\left(\frac{a}{2}\right)$

<u>Lemma</u>: Let $f: M_1 \to M_2$ be a diffeomorphism of class C^1 (and M_1, M_2 are connected). If, for every $p \in M$ and $v \in T_pM$, we have $\|df_p(v)\| = \|v\|$ then f is an isometry.

<u>Proof</u>: Let $p, q \in M_1$ and $\gamma : [a,b] \to M_1$ be C^1 curve such that $\gamma(a) = p$, $\gamma(b) = q$. Then $\tilde{\gamma} = f \circ \gamma$ be a curve in M_2 with $\tilde{\gamma}(a) = f(p)$ and $\tilde{\gamma}(b) = f(q)$. Moreover:

$$\ell(\tilde{\gamma}) = \int_a^b \|\dot{\tilde{\gamma}}\|\, dt = \int_a^b \|df_{\gamma(t)}(\dot{\gamma}(t))\|\, dt \overset{\text{Assumption}}{=} \int_a^b \|\dot{\gamma}(t)\|\, dt = \ell(\gamma)$$

So: $d_{M_2}(f(p), f(q)) \leq \ell(\tilde{\gamma}) = \ell(\gamma)$

By taking the infimum over such curves γ:

$$d_{M_2}(f(p), f(q)) \leq d_{M_1}(p,q).$$

Applying the same argument for $f^{-1}: M_2 \to M_1$, we also obtain the reverse inequality $\boxtimes$

In fact, the converse is also true:

Theorem (Myers-Steenrod): If $f: M_1 \to M_2$ is an isometry (for the intrinsic distances), then f is a C^1 map and, $\forall p \in M_1$, $df_p: T_pM_1 \to T_{f(p)}M_2$ is a (linear) isometry.

Definition: Two submanifolds $M_1, M_2 \subseteq \mathbb{R}^n$ are called congruent if there exists an isometry $T: \mathbb{R}^n \to \mathbb{R}^n$ with $T(M_1) = M_2$.

- In this case: $T|_{M_1}: M_1 \to M_2$ is an isometry.

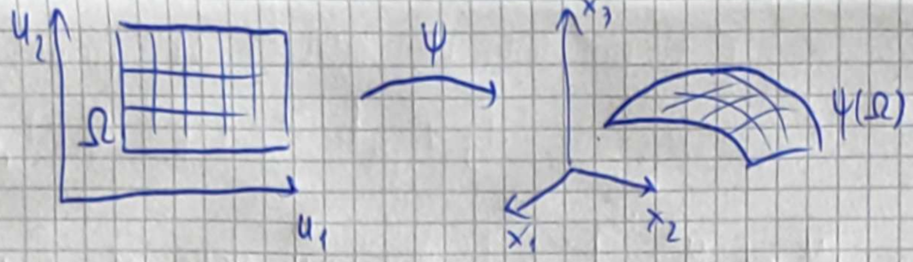
(But not every isometry of submanifolds can be extended to an ambient isometry. E.g. if $\gamma: \mathbb{R} \to \mathbb{R}^2$ is a naturally parametrized curve, then it is an isometry on it's image.)

The above suggests that understanding the metric properties of submanifolds, we need to first understand their "infinitesimal" geometry.

For this, we will need to work in local parametrizations of a submanifold by open domain in $\mathbb{R}^m$.

Let $M \subseteq \mathbb{R}^n$ be a submanifold of class C^k. Let

$\varphi: \Omega \subseteq \mathbb{R}^m \to M$ be a local parametrization (Ω is open, and $\varphi: \Omega \to \varphi(\Omega) \subseteq M$ is a diffeomorphism on its image).



Recall: $\psi : \Omega \longrightarrow \mathbb{R}^n$ is necessarily an <u>immersion</u>, namely

$d\psi_p : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is <u>injective</u> $\forall p \in \Omega$.

If $(u_1, \dots, u_m)$ are the Cartesian coordinates on Ω: This implies, in particular, that the vectors

$$b_1(u) = \frac{\partial \psi}{\partial u_1}(u), \quad \dots, \quad b_m(u) = \frac{\partial \psi}{\partial u_m}(u)$$

form a basis of $T_{\psi(u)}M$.

<u>Definition</u>: The <u>metric tensor</u> of M associated to the parametrization ψ is the $m \times m$ matrix valued function

$$G = (g_{ij}) : U \longrightarrow M_m(\mathbb{R})$$

given by $$g_{ij}(u) = \langle b_i(u), b_j(u) \rangle_{\mathbb{R}^n} = \sum_{k=1}^{n} \frac{\partial \psi_k}{\partial x_i} \cdot \frac{\partial \psi_k}{\partial x_j}$$

<u>Note</u>:
- If ψ is of class C^k, g is of class C^{k-1}
- (g_{ij}) is the matrix of the Euclidean inner product on $T_{\psi(u)}M \subseteq \mathbb{R}^n$ expressed in the basis $\{b_1, \dots, b_m\}$
- Another name for (g_{ij}): First fundamental form

So: If $v, w \in T_{\psi(u)}M$, $v = \sum_{i=1}^{m} v^i b_i$, $w = \sum_{j=1}^{m} w^j b_j$, then

$$\langle v, w \rangle = \sum_{i,j=1}^{m} g_{ij} v^i w^j$$

Examples: 1. If $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a C^1 function and S = it's graph, then a (global) parametrization for S:

$$\psi : \Omega \to \mathbb{R}^3, \quad \psi(x,y) = (x, y, f(x,y)).$$

Basis of $T_{\psi(x)} S$: $\quad b_1 = \frac{\partial \psi}{\partial x} = \begin{pmatrix} 1 \\ 0 \\ \partial_x f \end{pmatrix}, \quad b_2 = \frac{\partial \psi}{\partial y} = \begin{pmatrix} 0 \\ 1 \\ \partial_y f \end{pmatrix}$

So components of metric tensor:

$$g_{11} = \langle b_1, b_1 \rangle = 1 + (\partial_x f)^2, \quad g_{12} = \langle b_1, b_2 \rangle = \partial_x f \cdot \partial_y f, \quad g_{22} = 1 + (\partial_y f)^2$$

or, in matrix form, $G(x,y) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 + (\partial_x f)^2 & \partial_x f \cdot \partial_y f \\ \partial_x f \cdot \partial_y f & 1 + (\partial_y f)^2 \end{pmatrix}$

(Note: G always symmetric)

2. Surface of revolution

Let $a(v) = (r(v), z(v))$ be a regular curve of class C^1 $(v \in I \subseteq \mathbb{R})$

Assume that $r(v) > 0$ for $v \in I$

By rotating the curve around the z-axis: Surface of revolution, parametrized by $\psi : \Omega = [0, 2\pi] \times I \to S \subset \mathbb{R}^3$,

$$\psi(u,v) = \begin{pmatrix} r(v) \cdot \cos u \\ r(v) \sin u \\ z(v) \end{pmatrix}$$

u: longtitude, v: latitude

Basis of tangent plane:

$$b_1 = \frac{\partial \psi}{\partial u} = \begin{pmatrix} -r(v) \cdot \sin u \\ r(v) \cdot \cos u \\ 0 \end{pmatrix}, \quad b_2 = \frac{\partial \psi}{\partial v} = \begin{pmatrix} r'(v) \cdot \cos u \\ r'(v) \sin v \\ z'(v) \end{pmatrix}$$

Coefficients of metric tensor:

$$g_{11} = \left\| \frac{\partial \psi}{\partial u} \right\|^2 = (r(v))^2, \quad g_{12} = \left\langle \frac{\partial \psi}{\partial u}, \frac{\partial \psi}{\partial v} \right\rangle = 0, \quad g_{22} = \left\| \frac{\partial \psi}{\partial v} \right\|^2 = (r'(v))^2 + (z'(v))^2 = \|\dot{a}\|^2$$

So $G(u,v) = \begin{pmatrix} r(v)^2 & 0 \\ 0 & |\dot{\alpha}(v)|^2 \end{pmatrix}$

Note: Another way to rewrite the expression for the metric tensor.

$b_i = \frac{\partial \psi}{\partial x_i}$ is the i-th column of the Jacobian matrix

$$D\psi = \begin{bmatrix} \frac{\partial \psi_1}{\partial x_1} & \cdots & \frac{\partial \psi_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial \psi_n}{\partial x_1} & & \frac{\partial \psi_n}{\partial x_m} \end{bmatrix}$$

Then $g_{ij} = \sum_{k=1}^{n} \frac{\partial \psi_k}{\partial x_i} \frac{\partial \psi_k}{\partial x_j}$ is equivalent to $G = D\psi^T \cdot D\psi$. ①

Change of coordinate

If

$$\Omega_1 \xrightarrow{F} \Omega_2 \xrightarrow{\psi} M \subseteq \mathbb{R}^n$$
$$\Omega_1 \subseteq \mathbb{R}^m, \quad \Omega_2 \subseteq \mathbb{R}^m$$

$\psi : \Omega_2 \to M$ is a local paramentization, $F : \Omega_1 \to \Omega_2$ is a diffeomorphism then $\tilde{\psi} = \psi \circ F : \Omega_1 \to M$ is another local parametrization.

Relation between the metric tensors: If G the metric for ψ, $\tilde{G}$ for $\tilde{\psi}$, then:

$$\tilde{G}(p) = DF(p)^T \cdot G(F(p)) \cdot DF(p).$$

Proof: Composition rule: $\left.\frac{\partial \tilde{\psi}_k}{\partial x_i}\right|_p = \sum_{l=1}^{m} \left.\frac{\partial \psi_k}{\partial x_l}\right|_{F(p)} \left.\frac{\partial F_l}{\partial x_i}\right|_p$

So $D\tilde{\psi}|_p = D\psi|_{F(p)} \cdot DF|_p$

Then using ①, we obtain the result. ▨

Note: If $v \in T_pM$, then $\|v\|$ should be independent of the parametrization! (Check it!)

Geometric computation

If $\psi : \Omega \to M$ is a local parametrization of M, then

If $v, w \in T_{\psi(u)}M$, expressed in the basis $b_i = \frac{\partial \psi}{\partial x_i}$ as

$$v = \sum_{i=1}^{m} v^i b_i, \quad w = \sum_{j=1}^{m} w^j b_j \quad \text{, then}$$

$$\langle v, w \rangle = \sum_{i,j=1}^{m} g_{ij} v^i b^j$$

So (if $p = \psi(u)$), we have

$$\|v\| = \sqrt{\sum g_{ij} v^i v^j}$$ and, if θ is the angle between v, w:

$$\cos\theta = \frac{\sum g_{ij} v^i w^j}{\sqrt{\sum g_{ij} v^i v^j}\sqrt{\sum g_{ij} w^i w^j}}.$$

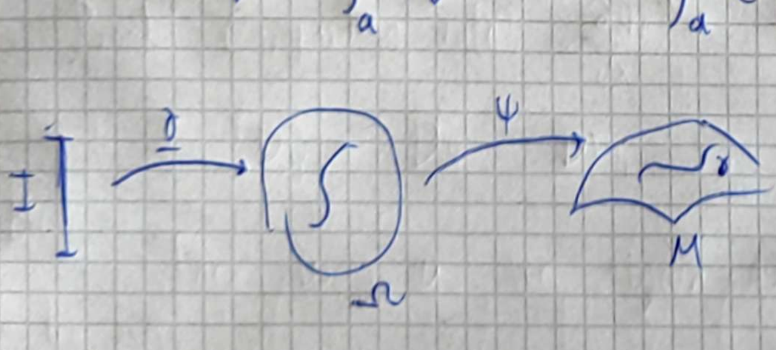
If $\gamma : I \to M$ is a C^1 curve:

Representation of γ in the parametrization:

$$\bar{\gamma}(t) = \psi^{-1}(\gamma(t)) = (u_1(t), \dots, u_m(t)) \in \Omega$$

And $$\dot{\gamma}(t) = \frac{d}{dt}\psi(\bar{\gamma}(t)) = \sum_{i=1}^{m} \frac{\partial \psi}{\partial u_i}\frac{du_i}{dt} = \sum_{i=1}^{m} \dot{u}_i b_i$$

So $$\ell(\gamma) = \int_a^b \|\dot{\gamma}(t)\|\, dt = \int_a^b \Big(\sum_{i,j=1}^{m} g_{ij}(u(t)) \cdot \dot{u}_i \dot{u}_j \Big)^{1/2} dt$$



Infinitesimally: If s the arc-length parameter, $\left(\frac{ds}{dt}\right)^2 = \|\dot{\gamma}\|^2 = \sum_{i,j} g_{ij} \frac{du_i}{dt}\frac{du_j}{dt}$

So schemmatically: Line element $$\boxed{ds^2 = \sum_{i,j=1}^{m} g_{ij}(u)\, du_i\, du_j}$$

The expression has the same content as the matrix G, but allows for easier manipulation.

For instance, if I change coordinates:

If $u_i = u_i(x_1, \dots, x_m)$, then $du_i = \sum_{k=1}^{m} \frac{\partial u_i}{\partial x_k} dx_k$.

So substituting for du_i in the line element, I get the expression of the metric in the $(x_1, \dots, x_m)$ coordinate system (same formula as before, but easier calculation in practice).

Also: If $\psi: \Omega \to \mathbb{R}^n$, setting $x_i = \psi_i(u)$ in $ds^2 = dx_1^2 + \dots + dx_n^2$ gives me the metric tensor.

E.g.: Line element of $\mathbb{R}^n$: $ds^2 = dx_1^2 + \dots + dx_n^2$

- Passing to polar coordinates (r, θ) in the plane:

$$x = r.\cos\theta, \quad y = r.\sin\theta$$

So

$$ds^2 = dx^2 + dy^2 = (\cos\theta \cdot dr - \sin\theta \cdot r \cdot d\theta)^2 + (\sin\theta \cdot dr + \cos\theta \cdot r \cdot d\theta)^2$$

$$= dr^2 + r^2 d\theta^2$$

So $G = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$: Euclidean metric in polar coordinates

- Back to the case of surface of revolution:

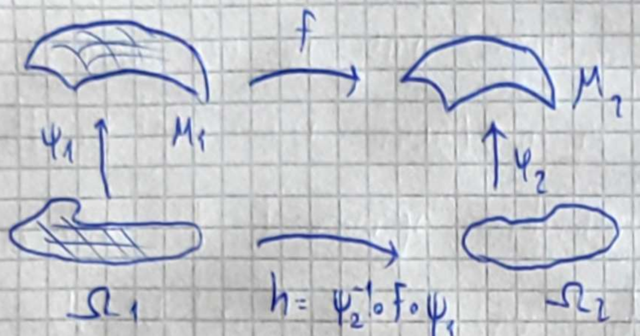
$$(u, v) \longrightarrow (x, y, z) = (r(v) \cdot \cos u, \ r(v) \cdot \sin u, \ z(v))$$

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$= (r'(v) \cdot \cos u \ dv - r(v) \cdot \sin u \ du)^2 + (r'(v) \sin u \cdot dv + r(v) \cos v \ du)^2 + (z'(v) \, dv)^2$$

$$= r(v)^2 du^2 + ((r'(v))^2 + (z'(v))^2) \, dv^2$$

(same as the formula we obtained before).



Theorem: Suppose that $\psi_1:\Omega_1\to M_1$ and $\psi_2:\Omega_2\to M_2$ are parametrizations of class C^1. Then there exists an isometry $f: M_1\to M_2$ if and only if there exists a diffeomorphism $h:\Omega_1\to\Omega_2$ such that

$$G_1(u) = Dh(u)^T\cdot G_2(h(u))\cdot Dh(u)$$

Remarks

- Note the similarity with the change of coordinates formula!
- h is the expression of f in the corresponding charts:

$$h=\psi_2^{-1}\circ f\circ\psi_1$$

- In components:

If $h(u_1,\dots,u_m)=(v_1,\dots,v_n)$, then

$$\underbrace{g_{ij}(u)}_{G_1}=\sum_{\alpha,\beta=1}^{n}\underbrace{\tilde g_{\alpha\beta}(v)}_{G_2}\,\frac{\partial v_\alpha}{\partial u_i}\,\frac{\partial v_\beta}{\partial u_j}$$

Proof:

Suppose that $f: M_1\to M_2$ is an isometry. Then

$\psi_2:\Omega_2\to M_2$ and $\tilde\psi_2=f\circ\psi_1:\Omega_1\to M_1$ are two (possibly different) parametrizations of M_2.

Let $p \in M_2$ $\left(p = \psi_2(\underset{\Omega_2}{\underset{\cap}{u}}) \text{ and } p = f\circ\psi_1(\underset{\Omega_1}{\underset{\cap}{x}}) \right)$.

Let $v \in T_pM_2$.

If $b_i = \frac{\partial \psi_2}{\partial x_i}$ is the basis w.r.t the first parametrization and

$\tilde{b}_i = \frac{\partial}{\partial x_i}(\underbrace{f\circ\psi_1}_{\tilde{\psi}_2}) = df\left(\frac{\partial \psi_1}{\partial x_i}\right)$ the basis w.r.t to the other one,

then

corresponding metric tensors:

$$g^{(2)}_{ij} = \left\langle \frac{\partial \psi_2}{\partial x_i}, \frac{\partial \psi_2}{\partial x_j} \right\rangle \quad \text{or} \quad G_2 = D\psi_2^T \cdot D\psi_2$$

and $$\tilde{g}^{(2)}_{ij} = \left\langle \frac{\partial \tilde{\psi}_2}{\partial x_i}, \frac{\partial \tilde{\psi}_2}{\partial x_j} \right\rangle$$

$$= \left\langle df\left(\frac{\partial \psi_1}{\partial x_i}\right), df\left(\frac{\partial \psi_1}{\partial x_j}\right) \right\rangle$$

Since df is an isometry: The above $= \left\langle \frac{\partial \psi_1}{\partial x_i}, \frac{\partial \psi_1}{\partial x_j} \right\rangle = g^{(1)}_{ij}$

So $\tilde{G}_2 = G_1$

If $h = \psi_2^{-1}\circ f \circ \psi_1$: Then $\tilde{\psi}_2 = \psi_2 \circ h$, so

$$G_1 = \tilde{G}_2 = D\tilde{\psi}_2^T \cdot D\tilde{\psi}_2 = Dh^T D\psi_2^T D\psi_2 Dh = Dh^T G_2 Dh.$$

The converse follows by arguing in the reverse direction.

<u>Corollaries</u>:

- If $\psi_1 : \Omega \to M_1$, $\psi_2 : \Omega \to M_2$ satisfies $G_1 = G_2$ then M_1 and M_2 are isometric
- If $\psi : \Omega_1 \to M$ and $\psi_2 : \Omega_2 \to M$ are parametrizations of the same submanifold: $\exists$ diff. $h : \Omega_1 \to \Omega_2$ such that $G_1 = Dh^T \cdot G_2 \cdot Dh$

Integration on submanifolds:

Def. Let $M \subsetneq \mathbb{R}^n$ be a submanifold of dimension m

and $\psi : \Omega \subsetneq \mathbb{R}^m \to M$ a C^1 global parametrization.

- $\mathrm{Vol}_m(M) = \int_\Omega \sqrt{\det G(u)}\, du_1 \cdots du_m$
- If ~~f~~ $f: M \to \mathbb{R}$: $\int_M f \, d\mathrm{vol} := \int_\Omega f \circ \psi(u) \sqrt{\det G(u)}\, du_1 \cdots du_m$

Remarks: If M not covered by a global parameter:

If $\psi_i : \Omega_i \to M$ such that $\psi_i(\Omega_i) \cap \psi_j(\Omega_j)$ have m-measure 0:

We can sum over the corresponding parametrizations

Exercise: Change of variables formula:

If $h : \Omega_1 \to \Omega_2$ diffeo, $\psi_1 = \psi_2 \circ h$

then $\int_{\Omega_1} \sqrt{\det G_1(u)}\, du = \int_{\Omega_2} \sqrt{\det G_2(v)}\, dv$

Use: $dv = \det(Dh)\, du$ and $G_1 = Dh^T G_2 Dh$. So: Geometric notion (independent of parametrization)

Example: For the graph of $f: \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$,

$$\mathrm{Area}(S) = \iint_\Omega \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}\, dx\, dy$$